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EQUIVALENCE OF ALGEBRAIC K-THEORIES

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0. Introduction

At the August 1972 Battelle conference on algebraic K-theory essentially four methods had been proposed for defining the higher K-groups $K_*(A)$ of a ring A with identity. These theories were K_*^Q of Quillen [8, 9]; K_*^V of Volodin [12]; K_*^S of Swan [11] with alternative approaches given by Gersten [5] and by Keune [7]; and K_*^{K-V} of Karoubi–Villamayor [6]. For a general survey, see [5]. There is a sequence of natural homomorphism [1]

$$K_*^Q \to K_*^V \to K_*^S \to K_*^{K-V}$$

and it turns out that the first three theories are naturally isomorphic for all A while all four are the same for A left regular. The purpose of this paper is to prove that $K_*^Q \cong K_*^V$; or more precisely, to show that $K_*^Q \cong K_*^{BN}$ where K_*^{BN} is a reformulation [13] of K_*^V related to the theory of BN-pairs. There are homomorphisms [2]

$$K_*^Q \to K_*^{BN} \to K_*^V$$

and the methods of this paper can probably be altered without difficulty to show that $K_*^Q \cong K_*^V$. For information concerning the other equivalences, see [1, 2, 5, 7].

Recall from [8] and [13] that for $i \geq 1$

$$K_i^Q(A) = \pi_i \, BGL(A)^+$$

and

$$K_i^{BN}(A) = \pi_{i-1} GL^{BN}(A).$$

In [2] it was shown that there is a homotopy fibration

$$GL^{BN}(A) + B\{U_F\}^+ \to BGL(A)^+$$

and we shall show the equivalence between K_*^Q and K_*^{BN} by proving as announced in [2] that

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Theorem I. *For any associative ring with identity A,*

$$\mathrm{GL}^{\mathrm{BN}}(A) \cong \Omega\, \mathrm{BGL}(A)^{+}.$$

This follows immediately from

Theorem II. $\tilde{H}_*(B\{U_F\}) = 0$ *and hence* $B\{U_F\}^+$ *is contractible.*

The method for proving this is inspired by Quillen's first proof that K_*^Q defined by the "plus construction" is equivalent to K_*^Q coming from the "Q-construction". It is a pleasure to thank D. Quillen for kindly allowing publication of these techniques as the Appendix to this paper.

1. A spectral sequence for $H_*(B\{U_F\})$

Let $\{H_\alpha\}$ be the collection of hyperplanes in n-dimensional euclidean space $\mathbf{R}^n$ given by the condition $\alpha = 0$ where $\alpha = e_i - e_j$, $i \neq j$, is a linear root. Here e_i is the i^{th} coordinate function. This determines a stratification of $\mathbf{R}^n$ whose strata we call facettes F as in [3]. A facette of codimension k is a component of the complement in the union of the k-fold intersections of the H_α of the subset consisting of the union of the $(k+1)$-fold intersections. Let $\mathcal{P}^n$ be the set of facettes of $\mathbf{R}^n$ partially ordered by the condition that $F < G$ iff $F \subset \bar{G}$. We shall also let $\mathcal{P}^n$ denote the simplicial complex whose k-simplices are $(k+1)$-tuples $(F_0 < \cdots < F_k)$ where $F_i \in \mathcal{P}^n$. Then $\mathcal{P}^n$ is a piecewise linear triangulation of the $(n-1)$-ball. The stabilization map $\mathbf{R}^n \to \mathbf{R}^{n+1}$ given by

$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_n, x_n) \tag{1.1}$$

takes each facette F to a facette F' and preserves the relation "$<$". Thus we can consider $\mathcal{P}^n$ as a subset (or as a subcomplex) of $\mathcal{P}^{n+1}$ and we let $\mathcal{P}^\infty = \bigcup_n \mathcal{P}^n$.

If $F \in \mathcal{P}^n$, let $U_F \subset \mathrm{GL}(n, A)$ be the subgroup generated by the elementary matrices $e_{ij}(\lambda)$ where $e_i - e_j > 0$ on F and $\lambda \in A$. Note that if $F \in \mathcal{P}^\infty$ lies, say, in $\mathcal{P}^n$, then $U_F \subset \mathrm{GL}(\infty, A)$ is the direct limit of the groups $U_F \hookrightarrow U_{F'} \hookrightarrow U_{F''} \hookrightarrow \cdots$.

Now for $1 \leq n \leq \infty$, let $B(n)$ be the realization of the simplicial space which in dimension $k \geq 0$ is the disjoint union of the spaces

$$(F_0 < F_1 < \cdots < F_k) \times BU_{F_0}$$

where $F_i \in \mathcal{P}^n$. Then $B(\infty) = \lim_{n\to\infty} B(n)$ and by definition we have $B\{U_F\} = B(\infty)$.

Another way of describing facettes $F \in \mathcal{P}^n$ is by partitions of the set $\{e_1, \ldots, e_n\}$ of standard basis vectors for $\mathbf{R}^n$. We write

$$F = X_1 | X_2 | \cdots | X_r$$

to mean that the X_i are disjoint subsets whose union is $\{e_1, \ldots, e_n\}$ and that F is determined by the conditions

$$e_\mu - e_\nu = 0 \quad \text{if} \quad e_\mu, e_\nu \in X_i,$$

$$e_\mu - e_\nu > 0 \quad \text{if} \quad e_\mu \in X_i, \quad e_\nu \in X_j \quad \text{and} \quad i < j.$$

When $n = \infty$ we require that each X_i is finite for $1 \leq i < r$. If $F = X_1 | X_2 | \cdots | X_r$ is in $\mathcal{P}^m$, $G = Y_1 | Y_2 | \cdots | Y_s$ is in $\mathcal{P}^n$, and $m, n < \infty$, we define $F \oplus G$ in $\mathcal{P}^{m+n}$ by

$$F \oplus G = X_1 | X_2 | \cdots | X_r | Y_1' | Y_2' | \cdots | Y_s',$$

where Y_i' is just Y_i shifted by adding m to get a subset of $\{e_{m+1}, \ldots, e_{m+n}\}$. Note that $F_1 < G_1$ and $F_2 < G_2$ implies $F_1 \oplus F_2 < G_1 \oplus G_2$.

If $\Delta \in \mathcal{P}^n$ is the diagonal facette defined by setting all $e_i - e_j = 0$, then there is another stabilization map $\mathcal{P}^m \to \mathcal{P}^{m+n}$ given by

$$F \to F \oplus \Delta. \tag{1.2}$$

This is not quite the same one induced by n repetitions of (1.1) yielding $F \to F^{(n)}$. However, we have

$$F^{(n)} < F \oplus \Delta. \tag{1.3}$$

This implies that the two stabilization maps $B(m) \to B(m+n)$ are homotopic. Compare [10].

To describe the spectral sequence for $B\{U_r\}$ it will be convenient to use homology groups of a simplicial complex K with coefficients in a sheaf $\mathcal{A}$ of abelian groups over K. A sheaf $\mathcal{A}$ over K consists of an abelian group A_σ for each simplex σ of K together with a homomorphism $i_{\sigma\tau} : A_\tau \to A_\sigma$ whenever $\sigma < \tau$ such that if $\sigma < \tau < \gamma$ then $i_{\sigma\gamma} = i_{\sigma\tau} \cdot i_{\tau\gamma}$. The groups $H_*(\mathrm{K}; \mathcal{A})$ are defined to be the homology of the complex $C_*(\mathrm{K}; \mathcal{A})$ in which the boundary of a k-chain

$$x = \sum_\sigma a_\sigma \cdot \sigma \quad (a_\sigma \in A_\sigma)$$

is given by

$$x = \sum_\sigma \left(\sum_{\tau < \sigma} i_{\tau\sigma}(a_\sigma) \cdot [\sigma; \tau] \cdot \tau \right)$$

where $[\sigma; \tau]$ is the incidence number between the k-simplex σ and its $(k-1)$-face τ.

Let $\mathcal{A} = \{A_\sigma\}$ and $\mathcal{B} = \{B_\tau\}$ be sheaves over simplicial complexes K and L respectively. Any pair $\Phi = (f, \phi)$ where $f : \mathrm{K} \to L$ is a simplicial map and $\phi : \mathcal{A} \to \mathcal{B}$ is a coefficient homomorphism over f (i.e. for each simplex σ of K there is a homomorphism $\phi_\sigma : A_\sigma \to B_{f(\sigma)}$ compatible with the $i_{\sigma\tau}$) induces homomorphisms

$$\Phi_* : C_*(\mathrm{K}; \mathcal{A}) \to C_*(L; \mathcal{B})$$

and

$$\Phi_* : H_*(\mathrm{K}; \mathcal{A}) \to H_*(L; \mathcal{B}).$$

Now let k be any algebraically closed field. Let $\mathcal{H}_q$ be the sheaf over $\mathcal{P}^n$ defined by

$$\mathcal{H}_q(\sigma) = H_q(BU_{F_0} : k)$$

where $\sigma = (F_0 < \cdots < F_k)$. Strictly speaking, we should index $\mathcal{H}_q$ by n but will not do so to keep notation simple. For any subcomplex $L \subset \mathcal{P}^n$ we shall also let $\mathcal{H}_q$ denote $\mathcal{H}_q$ restricted to L.

In [10] Segal has constructed a spectral sequence for the homology of the geometric realization of a simplicial space. This gives, for each $1 \leq n \leq \infty$, a spectral sequence converging to $H_*(B(n); k)$ with

$$E^1_{p,q}(n) = C_p(\mathcal{P}^n; \mathcal{H}_q)$$

and

$$E^2_{p,q}(n) = H_p(\mathcal{P}^n; \mathcal{H}_q).$$

The stabilization map $B(m) \to B(n)$ induced by (1.1) for $m < n$ induces a map of spectral sequences

$$i_* : E^r_{p,q}(m) \to E^r_{p,q}(n) \quad (1 \leq r \leq \infty)$$

and moreover

$$E^r_{p,q}(\infty) = \lim_{n\to\infty} E^r_{p,q}(n).$$

Proposition 1.4. *Let $q > 0$. If $2m \cdot (q+1) \leq n$, then*

$$i_* : E^2_{p,q}(m) \to E^2_{p,q}(n)$$

is the zero map.

Proof of Theorem II from Proposition 1.4. To show the reduced homology $\tilde{H}_*(B\{U_F\}; Z)$ is zero it is sufficient to show it vanished for coefficients in any algebraically closed field k. Proposition 1.4 implies $E^2_{p,q}(\infty) = 0$ for $q > 0$. But $E^2_{p,0}(\infty) = H_p(\mathcal{P}^\infty; k) = 0$ for $p > 0$ because $\mathcal{P}^\infty$ is contractible. Hence $E^2_{p,q}(\infty) = 0$ for all $p > 0$ and q, and so $\tilde{H}_*(B\{U_F\}; k) = 0$.

It will be helpful for the reader to keep in mind the following argument (cf. Appendix) which, when suitably modified, leads to the proof of Proposition 1.4. Let $G \subset \mathrm{GL}(m+n, A)$ be the subgroup of matrices of the form

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

and let $T \subset G$ be the subgroup with $X = Z = \mathrm{I}$. Suppose that $1/l \in A$ for some integer $l > 1$. Then $i : H_*(T; Q) \to H_*(G; Q)$ is the zero map. To see this let $D = Z[1/l]$. Then D^* acts on T by conjugation:

$$r \cdot \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} = \left(\begin{array}{ccc|ccc} r & & & & 0 & \\ & \ddots & & & & \\ & & r & & & \\ \hline & 0 & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right) \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \left(\begin{array}{ccc|ccc} r^{-1} & & & & 0 & \\ & \ddots & & & & \\ & & r^{-1} & & & \\ \hline & 0 & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right)$$

$$= \begin{pmatrix} I & r \cdot Y \\ 0 & I \end{pmatrix}.$$

This shows T is a D-module and the action of D^* is just scalar multiplication. Since T is an abelian group $H_q(T; Q) = \Lambda^q(T \otimes_Z Q)$ and hence $r \in D^*$ acts on the Q-vector space $H_q(T; Q)$ as scalar multiplication by r^q. But conjugation by r in G as above is trivial on $H_q(G; Q)$. So for $x \in H_q(T; Q)$ we have

$$i(x) = l \cdot i(x) = i(l \cdot x) = i(l^q x) = l^q i(x).$$

Thus $(l^q - 1)i(x) = 0$ which implies $i(x) = 0$.

Now to prove Proposition 1.4 for an arbitrary ring A the first step is to add enough units to A by enlarging it to a suitable $d \times d$ matrix ring $M_d(A)$ using Quillen's Lemma in the Appendix. The next step is to stabilize $x \in E^2_{p,q}(m)$ to $x \in E^2_{p,q}(m \cdot (q+1))$ so it can be pulled back to an element in the corresponding spectral sequence $E^2_{p,q}(m)$ for the ring $M_{q+1}(A)$. This is done in Sections 3 and 4. Finally we want to induce a non-trivial action on $E^2_{p,q}(m)$ via conjugation by diagonal matrices which is trivial when stabilized to $E^2_{p,q}(2m)$. This is Section 5. The problem is that the spaces $B(n)$ are built from the classifying spaces BU_F and the groups U_F contain only the identity diagonal matrix. This makes it impossible to say automatically (as in the above example) that conjugation on a homology class is trivial when pushed into a larger group. In Section 6 we put these three steps together with a triple induction argument to prove Proposition 1.4.

2. Adding units to A

Let B be any ring with identity and let G be a group. Let $\phi : G \to \mathrm{GL}(n, B)$ be a diagonal representation of the form $\phi(g) = \mathrm{diag}(\phi_1(g), \ldots, \phi_n(g))$ where each $\phi_i : G \to B^*$ is a homomorphism. If $F \in \mathcal{P}^n$, U_F is stable under conjugation by the diagonal subgroup $\phi(G)$ and we get an action of G on $H_q(BU_F)$ which will be called a *diagonal representation.*

Lemma 2.1. *Let A be any ring, $F \in \mathcal{P}^n$, $0 < q < d$, and let k be any algebraically closed field. Let $B = M_d(A)$ and $U_F \subset \mathrm{GL}(n, B)$ be the subgroup corresponding to F. Then there is an order D in a number field of degree d over Q, a filtration*

$$0 = W_0 \subset W_1 \subset \cdots \subset W_r = H_q(BU_F; k),$$

and for each $1 \leq i \leq r$ *a diagonal representation* ρ_i *of* D^* *on* $H_q(BU_F; k)$ *such that*

(a) *each* W_i *is stable under* ρ_i, *and*

(b) *the induced representation of* ρ_i *on* W_i/W_{i-1} *is the sum of non-trivial characters* $D^* \to k^*$.

The proof is a straightforward application of the Hochschild–Serre spectral sequence of a group extension with the key ingredient being Quillen's Lemma of the Appendix.

Proof of Lemma 2.1. Let D be as in Quillen's Lemma. Choosing a basis for D gives an imbedding $D \to M_d(Z)$ and following this by the homomorphism $M_d(Z) \to M_d(A)$ gives a homomorphism $\rho : D \to B$. Now write $F = X_1|X_2|\cdots|X_s$ and let $G = X_2|X_3|\cdots|X_s$. Then there is a split exact sequence

$$(*) \qquad 1 \to N \to U_F \to U_G \to 1$$

where N is abelian. Consider the diagonal representation $\rho_1 : D^* \to \mathrm{GL}(n, B)$ where $\rho_1(g)e_i = \rho(g)e_i$ for $e_i \in X_1$ and $\rho_1(g)e_i = e_i$ for $e_i \notin X_1$. Conjugation by $\rho_1(g)$ gives an automorphism of the exact sequence $(*)$ and hence an automorphism of the spectral sequence

$$E^2_{\mu,\nu} = H_\mu(U_G; H_\nu(N; k)) \Longrightarrow H_q(U_F; k)$$

where $\mu + \nu = q$. The representation of D^* on N is the restriction of the natural D-module structure coming from the conjugation automorphism. Hence for $\nu > 0$, $H_\nu(N; k)$ is the sum of non-trivial characters by Quillen's Lemma. Since the action of D^* on U_G is trivial, $E^\infty_{\mu,\nu}$ is also the sum of non-trivial characters for $\nu > 0$. By naturality of the spectral sequence we have a D^*-invariant filtration

$$0 \subset H_{0,q} \subset H_{1,q-1} \subset \cdots \subset H_{q-1,1} \subset H_{q,0} = H_q(U_F; k)$$

of the diagonal action ρ_1 on $H_q(U_F; k)$ such that $H_{\mu,q-\mu}/H_{\mu-1,q-\mu+1} \cong E^\infty_{\mu,q-\mu}$ is the sum of non-trivial characters for $0 \leq \mu < q$. Since the splitting inclusion $U_G \to U_F$ commutes with the conjugation by the diagonal matrices $\rho_1(g)$ we get a D^*-invariant decomposition

$$H_q(U_F; k) = H_{q-1,1} \oplus H_q(U_G; k)$$

where the action on $H_q(U_G; k)$ is trivial. Now let $\rho_2 : D^* \to \mathrm{GL}(n, B)$ be the diagonal representation "concentrated" at X_2; that is, $\rho_2(g)e_i = \rho(g)e_i$ for $e_i \in X_2$ and is the identity otherwise. As above we get a D^*-invariant filtration

$$0 \subset H'_{0,q} \subset H'_{1,q-1} \subset \cdots \subset H'_{q-1,1} \subset H'_{q,0} = H_q(U_G; k)$$

with $H'_{\mu,q-\mu}/H'_{\mu-1,q-\mu+1}$ the sum of non-trivial characters when $0 \leq \mu < q$ and also a D^*-invariant decomposition

$$H_q(U_G; k) = H'_{q-1,1} \oplus H_q(U_L; k)$$

where $L = X_3|X_4|\cdots|X_s$. D^* acts trivially on $H_q(U_L;k)$. Since conjugation by the matrices $\rho_2(g)$ is an automorphism of the exact sequence (*), the induced diagonal action of D^* on $H_q(U_F;k)$ leaves the subspaces $H_{\mu,q-\mu}$ invariant. Hence we can extend the filtration to one

$$0 \subset H_{0,q} \subset \cdots \subset H_{q-1,1} \subset H_{q-1,1} \oplus H'_{0,q} \subset \cdots \subset H_{q-1,1} \oplus H'_{q-1,1} \subset H_q(U_F;k)$$

which is invariant under both ρ_1 and ρ_2 and has successive quotients the sum of non-trivial characters except at the last stage where we get the identity on $H_q(U_L;k)$. Continuing in this way gives the desired result.

3. Elementary properties of sheaf homology

In this section we first prove some general lemmas which will be used later to show that certain elements of $C_*(\mathcal{P}^n;\mathcal{H}_q)$ are homologous. Then we show how the hypotheses of these lemmas are fulfilled in our situation.

Let K be a simplicial complex and $\mathcal{A} = \{A_\sigma\}$ a sheaf of abelian groups over K. Let there be given a triangulation of $\mathrm{K}\times\mathrm{I}$ which is a subdivision of the standard triangulation such that the induced triangulation of $\mathrm{K}\times 0$ is a copy of K. Define the sheaf $\mathcal{A}\times\mathrm{I} = \{A'_\tau\}$ over $\mathrm{K}\times\mathrm{I}$ by setting $A'_\tau = A_\sigma$ where σ is the smallest simplex of $\mathrm{K}\times 0$ such that $\tau\subset\sigma\times\mathrm{I}$. If $\tau<\tau'$ in $\mathrm{K}\times\mathrm{I}$ and $A'_\tau = A_\sigma$ and $A'_{\tau'} = A_{\sigma'}$, let $i_{\tau\tau'} = i_{\sigma\sigma'}$. Let $\mathcal{A}\times i$ be $\mathcal{A}\times\mathrm{I}$ restricted to $\mathrm{K}\times i$ for $i=0,1$. $\mathcal{A}\times 0$ is just $\mathcal{A}$. If $\mathrm{K}\times 1$ is a copy of K, then $\mathcal{A}\times 1$ is also just $\mathcal{A}$. In general, $\mathcal{A}\times 1$ is a "subdivision" of $\mathcal{A}$.

Now suppose we are given a pair $\Phi = (f,\phi)$ where $\phi:\mathcal{A}\to\mathcal{H}_q$ is a coefficient homomorphism over the simplicial map $f:\mathrm{K}\to\mathcal{P}^n$. Assume

(*) $\Omega = (w,\omega)$ is an extension of Φ where $\omega:\mathcal{A}\times\mathrm{I}\to\mathcal{H}_q$ covers $w:\mathrm{K}\times\mathrm{I}\to\mathcal{P}^n$ and $L\subset\mathcal{P}^n$ is a subcomplex such that $w(\mathrm{K}\times 1)\subset L$.

Let $\Psi = (g,\psi)$ be the restriction of Ω to $\mathrm{K}\times 1$ and $\mathcal{A}\times 1$ and let ρ denote the inclusion $L\to\mathcal{P}^n$ covered by $\mathcal{H}_q\,|\,L\to\mathcal{H}_q$.

Proposition 3.1. *Given a cycle* $x\in H_p(\mathrm{K}\times 0;\mathcal{A}\times 0)$ *there is a cycle* $x'\in H_p(\mathrm{K}\times 1;\mathcal{A}\times 1)$ *such that*

$$\Phi_*(x) = \rho_*\Psi_*(x')$$

in $H_p(\mathcal{P}^n;\mathcal{H}_q)$.

The proof shows that if $x = \sum a_\sigma\cdot\sigma$, where σ is a p-simplex of $\mathrm{K}\times 0$, then

$$x' = \sum_\sigma\Big(\sum_{\tau<\sigma\times 1}\varepsilon_\tau a_\sigma\cdot\tau\Big)$$

where τ is a p-simplex of $K \times 1$ and $\varepsilon_\tau = \pm 1$ depends only on τ. Proposition (3.1) is an immediate consequence of the following.

Lemma 3.2. *Let* K_p *denote the p-skeleton of* K. *Given* $x \in C_p(K_p \times 0; \mathscr{A} \times 0)$ *with* $\partial x = 0$, *there is exactly one* $y \in C_{p+1}(K_p \times I; \mathscr{A} \times I)$ *with* $\partial y = x + x'$ *where* $x' \in C_p(K_p \times 1; \mathscr{A} \times 1)$.

Proof. First consider uniqueness. Let $y_\sigma = y \,|\, \sigma \times I$. Then y_σ is an ordinary $(p+1)$-chain with coefficients in A_σ. Let τ and τ' be $(p+1)$-simplices in $\sigma \times I$ with a common open p-face $\tau \cap \tau'$. Note that $\tau \cap \tau' \subset \operatorname{int}(\sigma \times I)$. Since $\partial y = 0$ we know ∂y_σ evaluated on $\tau \cap \tau'$ is zero. Since each such open p-simplex $\tau \cap \tau' \subset \operatorname{int}(\sigma \times I)$ is the face of exactly two $(p+1)$-simplices, the value of y_σ on τ determines the value on τ' and vice versa. Since any two $(p+1)$-simplices τ and τ' of $\sigma \times I$ can be joined by a chain of $(p+1)$-simplices $\tau = \tau_1, \tau_2, \ldots, \tau_m = \tau'$ with $\tau_i \cap \tau_{i+1}$ an open p-simplex in $\operatorname{int}(\sigma \times I)$, the chain y_σ is determined by its value on the $(p+1)$-simplex τ with $\tau \cap (K_p \times 0) = \sigma$. But $\partial y = x + x'$ implies that

$$y_\sigma(\tau) = x(\sigma) \cdot [\tau; \sigma]$$

where $[\tau; \sigma]$ is the incidence number between τ and σ. Hence, the value of y on $\sigma \times I$ is uniquely determined by the value of x on σ.

Now existence: Let $x = \sum a_\sigma \cdot \sigma$ where σ runs over the p-simplices of K. For each p-simplex σ of K_p we have $H_*(\sigma \times I, \partial\sigma \times I \cup \sigma \times 1; A_\sigma) = 0$. Hence there is a unique $(p+1)$-chain y_σ on $\sigma \times I$ with values in A_σ such that $\partial y_\sigma = a_\sigma \cdot \sigma$ considered as a p-cycle in $H_p(\sigma \times I, \partial\sigma \times I \cup \sigma \times 1; A_\sigma)$. Define $y \in C_{p+1}(K_p \times I; \mathscr{A} \times I)$ to be y_σ on $\sigma \times I$. We must show that $\partial y = x + x'$ as desired. Clearly $\partial y \,|\, K_p \times 0 = x$. Let $x' = y \,|\, K_p \times 1$. Then it suffices to show that the value of ∂y on any p-cell τ of $K_p \times I$ is zero provided τ is not in $K_p \times 0$ or $K_p \times 1$. By the construction of y this is true for any $\tau \subset (\operatorname{int} \sigma) \times I$. It remains to consider the case where τ is a p-simplex of $\alpha \times I$ for some $(p-1)$-simplex α of $K_p \times 0$. Since $\partial y = 0$ the uniqueness argument implies it is sufficient to show $\partial y = 0$ on τ where $\tau \cap K_p \times 0 = \alpha$. Let

$$X = \text{star of } \alpha \text{ in } K_p \times 0,$$

$$X' = \text{link of } \alpha \text{ in } K_p \times 0,$$

$$Y = \text{star of } \alpha \text{ in } K_p \times I,$$

$$Y' = \text{link of } \alpha \text{ in } K_p \times I.$$

The value of ∂y on τ is computed by first restricting y to be a chain in $C_{p+1}(Y, Y'; \mathscr{A} \,|\, Y)$, and then mapping y into $C_{p+1}(Y, Y'; A_\alpha)$ by the coefficient homomorphism $\mathscr{A} \,|\, Y \to A_\alpha$ where A_α is the constant sheaf over Y. Now Y and Y' are contractible so $H_p(Y, Y'; A_\alpha) = 0$. Hence if x is considered as a cycle in $H_p(X, X'; A_\alpha)$ there is a unique chain $z \in C_{p+1}(Y, Y'; A_\alpha)$ such that $\partial z = x$. In particular $\partial z = 0$ on τ. Since $\partial y = 0$ on each p-simplex in int $\sigma \times \mathrm{I}$ for any p-simplex σ of K_p, the uniqueness argument implies $y = z$. Hence $y = \partial z = 0$ on τ.

Here is how condition (*) arises. We shall start with $\mathrm{K} = \mathscr{P}^m$ over which there is the sheaf $\mathscr{G}_m$ of non-abelian groups defined by

$$G_\sigma = U_{F_0}$$

for $\sigma = (F_0 < \cdots < F_n)$. There will be given a pair $\Phi = (f, \phi)$ where $f : \mathscr{P}^m \to \mathscr{P}^n$ is a map of partially ordered sets and ϕ is a homomorphism $\mathscr{G}_m \to \mathscr{G}_n$ over f. There will also be given an extension $w : \mathscr{P}^m \times \mathrm{I} \to \mathscr{P}^n$ of f where $\mathscr{P}^m \times \mathrm{I}$ is triangulated as an ordered simplicial complex subdividing the standard triangulation with $\mathscr{P}^m \times 0 = \mathscr{P}^m$ satisfying the following condition:

For each simplex σ of $\mathscr{P}^m$, and any vertex v of the subcomplex $\sigma \times \mathrm{I}$, we have

(**) $$\phi_\sigma(G_\sigma) \subset U_{w(v)}$$

as subgroups of $\mathrm{GL}(n, A)$. Moreover if v is any vertex of $\mathscr{P}^m \times 1$, then $w(v)$ is a vertex of a given subcomplex L of $\mathscr{P}^n$.

The cofficient homomorphism ϕ of abelian sheaves over f is obtained by taking the induced homology homomorphism

$$\phi_* : H_q(G_\sigma; k) \to H_q(U_{f(\sigma)}; k).$$

Define the coefficient homomorphism $\omega : \mathscr{H}_q \times \mathrm{I} \to \mathscr{H}_q$ over $w : \mathscr{P}^m \times \mathrm{I} \to \mathscr{P}^n$ as follows: Let $\tau = (v_0 < \cdots < v_k)$ be a simplex of $\mathscr{P}^m \times \mathrm{I}$ and $\sigma = (F_0 < \cdots < F_l)$ be the smallest simplex of $\mathscr{P}^m$ with $\tau \subset \sigma \times \mathrm{I}$. Then

$$(\mathscr{H}_q \times \mathrm{I})_\tau = G_\sigma = U_{F_0}$$

and $\omega_\tau : (\mathscr{H}_q \times \mathrm{I})_\tau \to (\mathscr{H}_q)_{w(\tau)}$ is the homology map

$$\phi_\sigma : H_q(G_\sigma; k) \to H_q(U_{w(v_0)}; k)$$

which exists in view of (**).

Next we explain how (**) will be obtained. Let $f : \mathrm{K} \to \mathscr{P}^n$ be a map of partially ordered sets as above. There are two cases:

Case 1. There will be given a map $g : \mathrm{K} \to \mathscr{P}^n$ of partially ordered sets such that for each facette F of K, $f(F) < g(F)$. Then for the standard triangulation of $\mathrm{K} \times \mathrm{I}$ there is clearly a map $w : \mathrm{K} \times \mathrm{I} \to \mathscr{P}^n$ satisfying (**).

Case 2. This is more complicated and here we just give the lemma that is used in the proof of (4.2) below.

Consider $f : \mathrm{K} \to \mathscr{P}^n$ as a map of partially ordered sets. Thus for each vertex v of K, $f(v)$ is a facette of $\mathbb{R}^n$. Now let $g : \mathrm{K} \to \mathscr{P}^n$ be any map (not necessarily one preserving the partial ordering). Give $\mathrm{K} \times \mathrm{I}$ the standard triangulation as a partially ordered set. The maps f and g define a map {vertices of $K \times I\} \to \mathcal{P}^n$ which doesn't necessarily preserve the order (except on $K = K \times 0$).

Lemma 3.3. *There is a triangulation $(K \times I)'$ of $K \times I$ as a partially ordered set which refines the standard triangulation leaving $K = K \times 0$ unchanged, and there is an order preserving map $w : (K \times I)' \to \mathcal{P}^n$ of this new triangulation such that*

(a) $w \mid K \times 0 = f$,

(b) *if v is a vertex of the new triangulation of $K \times 1$ which is also a vertex K, then*

$$w(v) = g(v),$$

(c) *if $\sigma = (v_0 < \cdots < v_k)$ is a simplex of the standard triangulation of $K \times I$, $v \in \sigma$ is a vertex of the new triangulation, and $e_{ij}(\lambda)$ is in $U_{w(v_s)}$ for $0 \leq s \leq k$, then*

$$e_{ij}(\lambda) \in U_{w(v)},$$

(d) *if $g : K \to \mathcal{P}^n$ is order preserving, then $K \times 1$ with the new triangulation is just a copy of K.*

Proof. Take $K \times I$ with the standard triangulation. For each vertex v of K choose a point in $f(v)$ and a point in $g(v)$. This choice defines a map from the vertices of $K \times I$ into $\mathbf{R}^n$. Extending linearly to each simplex we get a map

$$u : K \times I \to \mathbf{R}^n.$$

Consider the partition of $K \times I$ into the disjoint subsets $u^{-1}(F)$ where F is a facette of $\mathbf{R}^n$. Since u is a linear on each simplex we can find a refinement, say L, of the standard triangulation of $K \times I$ such that each (open) simplex of L is contained in some $u^{-1}(F)$. Since $f : K \to \mathcal{P}^n$ preserves order, L can be chosen so that $K = K \times 0$ remains unchanged; furthermore, the image of each open simplex of K under u is contained in a facette F. Now define the new triangulation of $K \times I$ to be the realization of the partially ordered set whose vertices are either the original vertices of K or an (open) simplex of $L - K$ equipped with the partial ordering which sets $\sigma < \tau$ iff

(i) σ and τ are vertices of K and $\sigma < \tau$ in K;

(ii) σ is a vertex of K, τ is an (open) simplex of $L - K$, and σ is a vertex of τ; or

(iii) σ and τ are (open) simplices of $L - K$ and σ is a face of τ.

Finally, define the order preserving map $w : K \times I \to \mathcal{P}^n$ by letting $w \mid K \times 0 = f$ and for each open simplex σ of $L - K$ letting $w(\sigma)$ be the unique facette F of $\mathbf{R}^n$ such that $u(\sigma) \subset F$.

Properties (a) and (b) are clearly satisfied. If $g : K \to \mathcal{P}^n$ is order preserving a suitable modification of (i) through (iii) gives (d). Regarding (c), the map w was constructed from u in such a way that for each vertex v of the original triangulation of $K \times I$ the point $u(v)$ of $\mathbf{R}^n$ belongs to the facette $w(v)$ of $\mathcal{P}^n$. Furthermore for each vertex $v \in \sigma$ the facette $w(v)$ has a non-empty intersection with the set $\sum_{s=0}^k t_s \cdot u(v_s)$ where $t_s \geq 0$ and $\sum_{s=0}^k t_k = 1$. By hypothesis $e_{ij}(\lambda) \in U_{w(v_s)}$ for each s;

that is, $e_i - e_j > 0$ on $w(v_s)$. Since $u(v_s) \in w(v_s)$, $e_i - e_j > 0$ on the point $\Sigma_s t_s \cdot u(v_s)$ and therefore $e_i - e_j > 0$ on the facette $w(v)$. This means $e_{ij}(\lambda) \in U_{w(v)}$.

Let $\phi, \phi' : \mathscr{A} \to \mathscr{H}_q$ be coefficient homomorphisms over the same order preserving map $f : \mathrm{K} \to \mathscr{P}^n$. Suppose there is an extension $w : \mathrm{K} \times \mathrm{I} \to \mathscr{P}^n$, say as in (3.3), where $\mathrm{K} \times \mathrm{I}$ has a triangulation as the nerve of a partially ordered set which refines the standard triangulation and has $\mathrm{K} = \mathrm{K} \times 0$. Suppose there are extensions $\omega, \omega' : \mathscr{A} \times \mathrm{I} \to \mathscr{H}_q$ of ϕ, ϕ' over w such that

$$(***) \qquad \omega \,|\, \mathscr{A} \times 1 = \omega' \,|\, \mathscr{A} \times 1.$$

Let $\Phi(f, \phi)$ and $\Phi' = (f, \phi')$.

Proposition 3.4. *If $x \in C_p(\mathrm{K}; \mathscr{A})$ is a cycle, then*

$$\Phi_*(x) = \Phi'_*(x)$$

in $H_p(\mathscr{P}^n; \mathscr{H}_q)$.

Proof. Let $y \in C_{p+1}(\mathrm{K} \times \mathrm{I}; \mathscr{A} \times \mathrm{I})$ be as in Lemma 3.2. Let $\Omega = (w, \omega)$ and $\Omega' = (w, \omega')$. Let $z = \Omega_*(y) - \Omega'_*(y)$, which lies in $C_{p+1}(\mathscr{P}^n; \mathscr{H}_q)$. Then

$$\begin{aligned} \partial z &= \Omega_*(x + x') - \Omega'_*(x + x') \\ &= \Omega_*(x) - \Omega'_*(x) + \Omega_*(x') - \Omega'_*(x') \\ &= \Omega_*(x) - \Omega'_*(x) \end{aligned}$$

by condition (***).

4. Stabilization

The two stabilization maps (1.1) and (1.2) of Section 1 induce two stabilization homomorphisms

$$E^r_{p,q}(m) \to E^r_{p,q}(n)$$

for $n \geq m + 2$. Since $F' < F \oplus \Delta$, (1.3) shows we are in Case 1 of Section 3 so that Proposition 3.1 implies these two maps are the same for $r \geq 2$. In this section we use the stabilization homomorphism induced by $F \to F \oplus \Delta$.

Suppose $n = md$ and write $\{e_1, \ldots, e_n\} = E_1 | E_2 | \cdots | E_m$ where $E_i = \{e_{(i-1)d+1}, \ldots, e_{id}\}$ for $1 \leq i \leq m$. Let $\mathscr{P}^{md}_d$ be the subcomplex of $\mathscr{P}^{md}$ whose vertices are facettes

$$F = X_1 | X_2 | \cdots | X_r$$

where each X_i is a union of some of the E_j's. For any ring A we have constructed in Section 1 spectral sequences $E^r_{p,q}(n)$. To make this clear we shall use the notation

$E^r_{p,q}(n,A)$. For each flag F of $\mathscr{P}^{md}_d$, the subgroup $U_F \subset \mathrm{GL}(n,A)$ can be considered as the subgroup $U_{\hat{F}} \subset \mathrm{GL}(m, M_d(A))$ for the appropriate facette $\hat{F} \in \mathscr{P}^m$. This gives isomorphisms

$$j_*: E^1_{p,q}(m, M_d(A)) \xrightarrow[\cong]{} C_p(\mathscr{P}^{md}_d; \mathscr{H}_q)$$

and

$$j_*: E^2_{p,q}(m, M_d(A)) \xrightarrow[\cong]{} H_p(\mathscr{P}^{md}_d; \mathscr{H}_q),$$

compatible with the stabilization maps. Moreover for $n = md$ and $n' = m'd$ with $m < m'$ there is a commutative diagram of spectral sequences

4.1.
$$\begin{array}{ccc} E^r_{p,q}(m, M_d(A)) & \xrightarrow{i_*} & E^r_{p,q}(m', M_d(A)) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ E^r_{p,q}(md, A) & \xrightarrow[i_*]{} & E^r_{p,q}(m'd, A). \end{array}$$

Let $\pi \in \mathrm{GL}(n,A)$ be a permutation matrix and let $\pi : \mathbf{R}^n \to \mathbf{R}^n$ be permutation of coordinates. This action of π takes facettes to facettes, preserves the ordering "$<$", and satisfies

$$\pi \cdot U_F \cdot \pi^{-1} = U_{\pi(F)}.$$

Hence π induces an automorphism $\pi : E^r_{p,q}(n') \to E^r_{p,q}(n')$ for $n \leq n'$.

Now let $n = md$, $d \geq 1$, and choose the permutation $\pi \in \mathrm{GL}(n,A)$ so that $\pi(i) \in E_i$ for $1 \leq i \leq m$.

Proposition 4.2. *Let $z \in E^2_{p,q}(m,A)$. Then there is a $z' \in E^2_{p,q}(m, M_d(A))$ such that*

$$\pi_* i_*(z) = j_*(z').$$

Proof. Let $\Phi = (f, \phi)$ where $f : \mathscr{P}^m \to \mathscr{P}^{md}$ is given by the correspondence $F \to F \oplus \Delta \to \pi(F \oplus \Delta)$ and ϕ is the coefficient homomorphism over f induced by taking homology of the group homomorphism

$$U_F \to U_{F\oplus\Delta} \to U_{\pi(F\oplus\Delta)}$$

where the second arrow is conjugation by π in $\mathrm{GL}(n,A)$. Then

$$\Phi_*(z) = \pi_* i_*(z) \in E^2_{p,q}(n,A).$$

Now let $g : \mathscr{P}^m \to \mathscr{P}^{md}$ be defined as follows: For any subset $X \subset \{1,\ldots,m\}$ let $X' \subset \{1,\ldots,md\}$ be given by

$$X' = \bigcup_{i \in X} E_i.$$

If $F \in \mathscr{P}^m$ is of the form $X_1|X_2|\cdots|X_r$, let $F' = X'_1|X'_2|\cdots|X'_r$. Note that the operation $F \to F'$ just defined is not stablilization as defined in Section 1. In this section we use the stabilization $F \to F \oplus \Delta$. The correspondence $F \to F'$ gives an order preserving map $g : \mathcal{P}^m \to \mathcal{P}^{md}$. Now use (3.3) to extend $f : \mathcal{P}^m \times 0 \to \mathcal{P}^{md}$ and $g : \mathcal{P}^m \times 1 \to \mathcal{P}^{md}$ to a map $w : \mathcal{P}^m \times \mathrm{I} \to \mathcal{P}^{md}$. We want to invoke (c) of (3.3) to extend the coefficient homomorphism ϕ over f to one $\omega : \mathcal{H}_q \times \mathrm{I} \to \mathcal{H}_q$ over w. So let $\sigma = (F_0 < \cdots < F_k)$ be any simplex of $\mathcal{P}^m$ and v be any vertex of a simplex in the standard triangulation of $\tau \times \mathrm{I}$. Then

$$w(v) = \pi(F \oplus \Delta),$$

or

$$w(v) = F',$$

where F is one of the facettes F_i. Let $\mathrm{e}_{ij}(\lambda) \in U_F$ be any generator. Then $\phi_\sigma(\mathrm{e}_{ij}(\lambda)) = \dot{\mathrm{e}}_{\pi(i),\pi(j)}(\lambda)$ lies in $U_{\pi(F\oplus\Delta)}$. Suppose $F = X_1|X_2|\cdots|X_r$. Then $\mathrm{e}_{ij}(\lambda) \in U_F$ means that $i \in X_a$ and $j \in X_b$ where $a < b$. But $\pi(i) \in X'_a$ and $\pi(j) \in X'_b$, which means $\phi_\sigma(\mathrm{e}_{ij}(\lambda)) = \mathrm{e}_{\pi(i),\pi(j)}(\lambda)$ also lies in $U_{F'}$. This shows that (c) of (3.3) is satisfied for each $\mathrm{e}_{\pi(i),\pi(j)}(\lambda)$ corresponding to a generator $\mathrm{e}_{ij}(\lambda)$ of U_F. Therefore (**) and hence (*) hold with $L = \mathcal{P}^{md}_d$. Letting $\Psi = (g, \psi)$ be the restriction of $\Omega = (w, \omega)$ as in (*), set

$$z' = \Psi_*(z) \in H_p(\mathcal{P}^{md}_d; \mathcal{H}_q).$$

Then by (3.1) we get $\pi_* i_*(z) = j_*(z')$.

5. Homological triviality of stable conjugation

Let $\alpha \in \mathrm{GL}(m, A)$ be a diagonal matrix. Then conjugation by α takes each subgroup U_F to itself and induces an automorphism of each spectral sequence $E^r_{p,q}(n)$ for $m \leq n$. If z is an element of $E^1_{p,q}(n) = C_p(\mathcal{P}^n; \mathcal{H}_q)$ or of $E^2_{p,q}(n) = H_p(\mathcal{P}^n; \mathcal{H}_q)$, we denote the action of α on z by $\alpha \cdot z$. The stabilization map $i_* : E^r_{p,q}(m) \to E^r_{p,q}(n)$ is equivariant: $i_*(\alpha \cdot z) = \alpha \cdot i_*(z)$.

Proposition 5.1. *Let $z \in H_p(\mathcal{P}^m; \mathcal{H}_q)$ and let $i_* : H_p(\mathcal{P}^m; \mathcal{H}_q) \to H_p(\mathcal{P}^{2m}; \mathcal{H}_q)$ be stabilization. Then $\alpha \cdot i_*(z) = i_*(z)$.*

Proof. Let

$$\beta = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

Then $\alpha \cdot i_*(z) = \beta \cdot i_*(z)$ and the proof of (5.1) will be based on the decomposition

$$(\dagger) \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\gamma^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{GL}(2m, A)$ where $\gamma = \alpha^{-1}$.

The map i_* is Ψ_* where $\Psi = (f, \phi_*)$ is the pair with $f : \mathcal{P}^m \to \mathcal{P}^{2m}$ defined by $f(F) = F \oplus \Delta$ and ϕ_* is the coefficient homomorphism obtained by taking homology of $\phi : U_F \to U_{F\oplus\Delta}$ given by

$$\phi(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\Psi^1 = (f^1, \phi^1_*)$ where $f^1 = f$ and $\phi^1 : U_F \to U_{F\oplus\Delta}$ is the coefficient homomorphism over f^1 defined by

$$\phi^1(x) = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \cdot \phi(x) \cdot \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x & \gamma - x\gamma \\ 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

lies in $U_{F\oplus\Delta}$ for each $F \in \mathcal{P}^m$ and conjugation is trivial on homology, we have $\phi_* = \phi^1_*$ and

$$\Psi_*(x) = \Psi^1_*(x).$$

Now let $f^2 : \mathcal{P}^m \to \mathcal{P}^{2m}$ be given by $f^2(F) = F \oplus F$. Since each matrix $\phi^1(x)$ lies in $U_{F\oplus\Delta}$ and in $U_{F\oplus F}$ for each $F \in \mathcal{P}^m$ and each $x \in U_F$, setting $\phi^2 = \phi^1$ gives a coefficient homomorphism ϕ^2_* over f^2. Furthermore, since $F \oplus \Delta < F \oplus F$ for each $F \in \mathcal{P}^m$, we can apply Case 1 of Section 3 plus Proposition 3.1 to get

$$\Psi^1_*(z) = \Psi^2_*(z)$$

where $\Psi^2 = (f^2, \phi^2_*)$.

For each $F = X_1 | X_2 | \cdots | X_r$ in $\mathcal{P}^m$, let $F \square F \in \mathcal{P}^{2m}$ be

$$F \square F = X_1' | X_1 | X_2' | X_2 | \cdots | X_r' | X_r$$

where for each subset $X \subset \{1, \ldots, m\}$ the subset $X' \subset \{m+1, \ldots, 2m\}$ is obtained by adding m to each element of X.

Consider the map $g : \mathcal{P}^m \to \mathcal{P}^{2m}$ defined by $g(F) = F \square F$. This is not order preserving! Apply Lemma 3.3 to find a new triangulation of $\mathcal{P}^m \times \mathrm{I}$ and an order preserving map $w : \mathcal{P}^m \times \mathrm{I} \to \mathcal{P}^{2m}$ as in (3.3). Now the point is that for each $x \in U_F$, $\phi^2(x)$ lies in $U_{F\oplus F}$ and in $U_{F\square F}$. Moreover if $F < G$, then $\phi^2(x)$ also is in $U_{G\oplus G}$ and in $U_{G\square G}$. This implies condition (c) of (3.3) holds, and so therefore (**) of Section 3 also holds. Thus we get the coefficient homomorphism $\omega : \mathcal{H}_q \times \mathrm{I} \to \mathcal{H}_q$ over w.

Now let $f^3 : \mathcal{P}^m \to \mathcal{P}^{2m}$ be just f^2. Let the coefficient homomorphism ϕ^3_* over f^3 be induced on homology by

$$\phi^3(x) = \begin{pmatrix} 1 & 0 \\ \gamma^{-1} & 1 \end{pmatrix} \cdot \phi^2(x) \cdot \begin{pmatrix} 1 & 0 \\ -\gamma^{-1} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} x & \gamma - x\gamma \\ 0 & \gamma^{-1}x\gamma \end{pmatrix}.$$

Since for each $F \in \mathcal{P}^m$ the matrix $\phi^3(x)$ lies both in $U_{F\oplus F}$ and in $U_{F\Box F}$, we get, as above, another coefficient homomorphism $w' : \mathcal{H}_q \times \mathrm{I} \to \mathcal{H}_q$ over w. Furthermore, for each $F \in \mathcal{P}^m$ the matrix

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

belongs to $U_{F\Box F}$ because $\gamma = \alpha^{-1}$ is a diagonal matrix! By condition (c) of (3.3) we therefore conclude that

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

lies in $U_{w(v)}$ for each vertex v of the triangulation induced on $\mathcal{P}^m \times 1$ by the new triangulation on $\mathcal{P}^m \times \mathrm{I}$. Since conjugation is trivial on group homology, we have

$$\omega \mid \mathcal{H}_q \times 1 = \omega' \mid \mathcal{H}_q \times 1.$$

That is, (***) of Section 3 is satisfied. Proposition 3.4 implies that

$$\Psi^2_*(z) = \Psi^3_*(z).$$

Now let $f^4 = f^3$ and ϕ^4_* be the coefficient homomorphism over f^4 induced by

$$\phi^4(x) = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \cdot \phi^3(x) \cdot \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1}x\gamma \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}\right) \in U_{F\oplus F}$ for each $F \in \mathcal{P}^m$ and conjugation on group homology is trvial, we get

$$\Psi^3_*(x) = \Psi^4_*(x)$$

where $\Psi^4 = (f^4, \phi^4_*)$.

Continuing in this way using the decomposition (†) shows that the map i_*, which is induced by the correspondence $U_F \to U_{F\oplus\Delta}$ given by

$$x \to \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

is the same as the map $\alpha \cdot i_*$, which is induced by the homomorphism

$$x \to \begin{pmatrix} \alpha x \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

6. Proof of Proposition 1.4

Let $z \in E^2_{p,q}(m, A)$. Let $B = M_{q+1}(A)$ and using (4.2) choose a $z' \in E^2_{p,q}(m, B)$ such that $\pi_* i_*(z) = j_*(z')$ in $E^2_{p,q}(m(q+1), A)$. Since the stabilization map $E^2_{p,q}(m(q+1), A) \to E^2_{p,q}(2m(q+1), A)$ is equivariant for the action π_*, the commutativity of the square (4.1) shows that to prove (1.4) it suffices to show

$$i_* : E^2_{p,q}(m, B) \to E^2_{p,q}(2m, B)$$

is the zero map.

Let $Z \subset C_p(\mathcal{P}^m; \mathcal{H}_q) = E^1_{p,q}(m, B)$ be the set of cycles and for $0 \leq s < \infty$ let $Z_s \subset Z$ be the subset of those cycles of the form $z = \sum a_\tau \cdot \tau$ where $a_\tau \in \mathcal{H}_q(\tau)$ is possibly non-zero for at most s p-cells τ. We prove (1.4) by showing that for each $s \geq 0$ the image of Z_s in $H_p(\mathcal{P}^{2m}, \mathcal{H}_q)$ is zero. Clearly this is true for $s = 0$ because $Z_0 = 0$.

First Induction Hypothesis. For all $0 \leq s' < s$ the image of $Z_{s'}$ in $H_p(\mathcal{P}^{2m}; \mathcal{H}_q)$ is zero.

Let $\sigma = (F < F_1 < \cdots < F_p)$ be a p-cell of $\mathcal{P}^m$ and let $z \in Z_s$ be a chain of the form

$$z = y + x_\sigma \cdot \sigma \tag{6.1}$$

where $x_\sigma \in H_q(BU_F; k)$ and $y \in C_p(\mathcal{P}^m; \mathcal{H}_q)$ is a chain with at most $s - 1$ non-zero terms. The chains $x = x_\sigma \cdot \sigma$ and y are not necessarily cycles.

Let $0 = W_0 < W_1 < \cdots < W_r = H_q(BU_F; k)$ be the filtration obtained in (2.1) where we take $d = q + 1$. We will show that any cycle z of the form (6.1) goes to zero in $H_p(\mathcal{P}^{2m}; \mathcal{H}_q)$ by showing inductively that for each $0 \leq a \leq r$, z goes to zero provided $x_\sigma \in W_a$. Since any chain of Z_s is of the form (6.1) for some p-cell σ, this will imply all of Z_s goes to zero. If $a = 0$, then $z \in Z_{s-1}$ and it goes to zero by the first induction hypothesis. So let $x_\sigma \in W_a$.

Second Induction Hypothesis. For $0 \leq a' < a$ any chain z of the form (6.1) goes to zero in $H_p(\mathcal{P}^{2m}; \mathcal{H}_q)$ provided $x_\sigma \in W_{a'}$.

As in (2.1) let D^* act on $H_q(BU_F; k)$ via a homomorphism $\theta : D^* \to \mathrm{GL}(m, B)$ into the subgroup of diagonal matrices followed by conjugation. As in Section 5 this gives a representation of D^* on the k-vector spaces $C_p(\mathcal{P}^n; \mathcal{H}_q)$ and $H_p(\mathcal{P}^n; \mathcal{H}_q)$ for $m \leq n$ which is compatible with stabilization. If $\eta \in D^*$ and z lies in $C_p(\mathcal{P}^n; \mathcal{H}_q)$ or $H_p(\mathcal{P}^n; \mathcal{H}_q)$, denote the action of η on z by $\eta \cdot z$. Note that if the chain z is written as a sum $\sum_\tau a_\tau \cdot \tau$ with $a_\tau \in \mathcal{H}_q(\tau)$ then $\eta \cdot z = \sum_\tau a'_\tau \cdot \tau$ with $a'_\sigma \in \mathcal{H}_q(\tau)$ differing from a_τ by the automorphism of $\mathcal{H}_q(\tau)$ induced by conjugation by $\theta(\eta)$. In particular, if z is of the form (6.1) so is $\eta \cdot z$.

Now suppose $x_\sigma \in W_a$. By (2.1) choose the representation of D^* so that on

W_a/W_{a-1} it is a sum $\bigoplus_\alpha V_\alpha$ of the eigenspaces V_α of distinct, non-trivial characters $\alpha : D^* \to k^*$.

Let $\pi : W_a \to W_a/W_{a-1}$ be the projection and for $b \geq 0$ let $Y_b \subset W_a$ be the D^*-invariant subset of those z of the form (6.1) such that $\pi(x_\sigma)$ has at most b non-zero components in $\bigoplus_\alpha V_\alpha$. We will show that any z of the form (6.1) with $x_\sigma \in W_a$ goes to zero by showing inductively over b that z goes to zero when $x_\sigma \in Y_b$. Clearly if $b = 0$, then $x_\sigma \in W_{a-1}$ and hence z goes to zero by the second induction hypothesis. So suppose $x_\sigma \in Y_b$.

Third Induction Hypothesis. For $0 \leq b' < b$ any chain z of the form (6.1) goes to zero provided $x_\sigma \in Y_{b'}$.

Let $\alpha : D^* \to k^*$ be one of the non-trivial characters such that $\pi(x_\sigma)$ has a non-zero component in V_α. Choose an $\eta \in D^*$ such that $\alpha(\eta) \neq 1$. Then $\eta \cdot x_\sigma = \alpha(\eta)x_\sigma + v$ where $\eta \cdot x_\sigma$ is the diagonal representation action of (2.1); $\alpha(\eta)x_\sigma$ is scalar multiplication in the k-vector space $H_q(BU_F; k)$; and, finally, $v = \eta \cdot x_\sigma - \alpha(\eta)x_\sigma$ lies in Y_{b-1}. The chain $\eta \cdot z - \alpha(\eta)z \in C_p(\mathcal{P}^m; \mathcal{H}_q)$ is a cycle and furthermore

$$\begin{aligned}\eta \cdot z - \alpha(\eta)z &= \eta \cdot y + (\eta \cdot x_\sigma)\cdot\sigma - \alpha(\eta)y - (\alpha(\eta)x_\sigma)\cdot\sigma \\ &= \{\eta \cdot y - \alpha(\eta)y\} + (\alpha(\eta)x_\sigma + v)\cdot\sigma - (\alpha(\eta)x_\sigma)\cdot\sigma \\ &= \{\eta \cdot y - \alpha(\eta)y\} + v\cdot\sigma.\end{aligned}$$

Since $v \in Y_{b-1}$, the third induction hypothesis implies that $\eta \cdot z - \alpha(\eta)z$ goes to zero in $H_p(\mathcal{P}^{2m}; \mathcal{H}_q)$. Hence

$$\begin{aligned}0 &= i_*(\eta \cdot z - \alpha(\eta)z) \\ &= i_*(\eta \cdot z) - i_*(\alpha(\eta)z) \\ &= \eta \cdot i_*(z) - \alpha(\eta)i_*(z) \\ &= i_*(z) - \alpha(\eta)i_*(z) \qquad\qquad \text{(by (5.1))} \\ &= (1-\alpha(\eta))i_*(z).\end{aligned}$$

Since $1 - \alpha(\eta) \neq 0$, $i_*(z) = 0$ in the k-vector space $H_p(\mathcal{P}^{2m}; \mathcal{H}_q)$.

Appendix*

Let A be an associative ring with identity. Let GL_n be the group of invertible $n \times n$ matrices over A and let $\mathrm{GL}_\infty = \lim_n \mathrm{GL}_n$. For $1 \leq m, n \leq \omega$ let

* By D. Quillen.

$$\mathrm{GL}_{m,n}=\left(\begin{array}{c|c}\mathrm{GL}_m & N\\ \hline 0 & \mathrm{GL}_n\end{array}\right)$$

where N is the set of $m \times n$ matrices over A. In case m or n is infinite, take N to be those matrices with only finitely many non-zero entries. There are homomorphisms

$$\mathrm{GL}_m \times \mathrm{GL}_n \xrightarrow[i_{m,n}]{} \mathrm{GL}_{m,n} \xrightarrow[\pi_{m,n}]{} \mathrm{GL}_m \times \mathrm{GL}_n$$

with $\pi_{m,n} \circ i_{m,n} = \text{identity}$.

Theorem A.1. *Let k be a field and assume there is a prime number l invertible in A which divides* char(k), *hence either* char(k) = l *or* char(k) = 0. *Then*

$$(\pi_{m,n})_* i H_*(\mathrm{GL}_{m,n}; k) \to H_*(\mathrm{GL}_m \times \mathrm{GL}_n; k)$$

is an isomorphism.

When char(k) = 0 this theorem applies to any ring in which some prime number is invertible. Let L be the set of prime numbers invertible in A. Assuming this is non-empty, the theorem says $\pi_{m,n}$ induces isomorphisms on homology with coefficients in Q and Z/l for all l in L. By standard universal coefficient arguments, it follows that $\pi_{m,n}$ induces isomorphisms on homology and cohomology with coefficients in any abelian group which is uniquely p-divisible for all primes p not in L. For example, $\pi_{m,n}$ induces isomorphisms on integral homology when A is an algebra over Q.

Let $i = i_{m,n}$ and $\pi = \pi_{m,n}$ when $m = n = \infty$.

Theorem A.2. *The homomorphism π_* is an isomorphism for homology with integral coefficients.*

Thus the subgroup $N = \ker \pi$ disappears for homology in the stable case. This is the algebraic analogue of the fact that N is contractible in the situation where, say, A is the real numbers and $\mathrm{GL}_{m,n}$ and $\mathrm{GL}_m \times \mathrm{GL}_n$ have the usual topology yielding a homotopy equivalence $\mathrm{BGL}_{m,n} \to \mathrm{BGL}_m \times \mathrm{BGL}_n$.

Proof of Theorem A.1. Consider the spectral sequence

$$E^2_{st} = H_s(\mathrm{GL}_m \times \mathrm{GL}_n; H_t(N; k)) \Longrightarrow H_{s+t}(\mathrm{GL}_{m,n}; k)$$

corresponding to the extension

$$0 \to N \to \mathrm{GL}_{m,n} \xrightarrow[\pi_{m,n}]{} \mathrm{GL}_m \times \mathrm{GL}_n \to 1.$$

The abelian group N is an A-module via scalar multiplication on the rows. Since l is invertible in A, N is also a module over $D = Z[l^{-1}]$. If char(k) = l, one has

$H_t(N;k)=0$ for $t>0$ and $H_0(N;k)=k$. The spectral sequence therefore degenerates giving the desired result.

If $\mathrm{char}(k)=0$, let $h: D^* \to \mathrm{GL}_m \times \mathrm{GL}_n$ be the homomorphism taking $u \in D^*$ to the matrix

$$\left[\begin{array}{ccc|c} u & & & \\ & \ddots & & 0 \\ & & u & \\ \hline & 0 & & I_n \end{array}\right].$$

Make D^* act on $\mathrm{GL}_{m,n}$ by conjugation: $u \cdot g = h(u) \cdot g \cdot h(u)^{-1}$. Then D^* acts trivially on $\mathrm{GL}_m \times \mathrm{GL}_n$ and on N it acts by multiplying with respect to the D-module structure. Thus an element u of D^* acts on

$$H_t(N;k)=\Lambda^t(N \otimes_Z k)$$

by multiplying by u^t, where we identify D with a subring of k in a unique way. Since D^* acts trivially on $\mathrm{GL}_m \times \mathrm{GL}_n$, u acts on E^2_{st} by multiplying by u^t. The differential commutes with the D^*-action, and as $u^t \neq u^{t'}$ for $t \neq t'$ when $u = l$, all differentials are zero. Finally, D^* acts trivially on the abutement, because inner automorphisms of a group induce the identity on its homology. Thus $E^2_{st}=0$ for $t>0$ and the spectral sequence degenerates giving the desired result.

Proof of Theorem A.2. To show π_* is an isomorphism with integer coefficients it suffices to show it is an isomorphism for coefficients in any algebraically closed field. So from now on let k denote a fixed, but arbitrary, algebraically closed field. It also suffices to show i_* is a surjection. For a moment we admit the following.

Lemma. *Let k be an algebraically closed field and d an integer >0. Then there exists an order D in a number field of degree d over Q with the following properties: Given any D-module N, let the group of units D^* act on it by multiplication, and let the group homology $H_*(N;k)$ be endowed with the induced action of D^*. Then for each t, $H_t(N;k)$ is a direct sum of one-dimensional representations of D^* over k. Furthermore, $H_t(N;k)$ does not contain the trivial representation for $0<t<d$.*

The direct sum operation which associates to a pair of matrices

$$M=\begin{pmatrix}\alpha & u \\ 0 & \beta\end{pmatrix} \quad \text{and} \quad M'=\begin{pmatrix}\alpha' & u' \\ 0 & \beta'\end{pmatrix}$$

the matrix

$$M \oplus M' = \begin{bmatrix} \alpha & 0 & u & 0 \\ 0 & \alpha' & 0 & u' \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta' \end{bmatrix},$$

is a group homomorphism $\mathrm{GL}_{\infty,\infty} \times \mathrm{GL}_{\infty,\infty} \to \mathrm{GL}_{\infty,\infty}$ which gives rise to a Hopf algebra structure on $H_*(\mathrm{GL}_{\infty,\infty}; k)$. Similarly $H_*(\mathrm{GL}_\infty \times \mathrm{GL}_\infty; k)$ is a Hopf algebra and the inclusion

$$i_*: H_*(\mathrm{GL}_\infty \times \mathrm{GL}_\infty; k) \to H_*(\mathrm{GL}_{\infty,\infty}; k)$$

is a Hopf algebra homomorphism. Assume inductively that i_* is surjective in degree less than n and let u be an element of $H_n(\mathrm{GL}_{\infty,\infty}; k)$. For p sufficiently large u pulls back to a class in $H_n(\mathrm{GL}_{p,p}; k)$ which we still call u. Choose $d > n$ and such that $d \neq 0$ in k. Let $A' = M_d(A)$ be the ring of $d \times d$ matrices over A. Consider A as a subring by imbedding it along the diagonal. This gives an imbedding

$$\Delta: \mathrm{GL}_{p,p}(A) \to \mathrm{GL}_{dp,dp}(A) = \mathrm{GL}_{p,p}(A')$$

which is conjugate to the d-fold direct sum

$$\oplus^d: \mathrm{GL}_{p,p}(A) \to \mathrm{GL}_{dp,dp}(A).$$

Hence $\Delta(u) = du + \Sigma_i v_i$ where each v_i is the product of terms of degree less than n. Since $d \neq 0$ in k it suffices by the inductive assumption to show $\Delta(u)$ lies in the image of $i_*: H_n(\mathrm{GL}_p(A') \times \mathrm{GL}_p(A'); k) \to H_n(\mathrm{GL}_{p,p}(A'); k)$.

Let D be as in the lemma and choose a basis over Z of d elements for D. This yields a homomorphism $\rho_0: D \to M_d(Z) \to M_d(A)$ and in turn a homomorphism $\rho: D^* \to \mathrm{GL}_{p,p}(A')$ where

$$\rho(\lambda) = \left[\begin{array}{ccc|c} \rho_0(\lambda) & & & \\ & \ddots & & 0 \\ & & \rho_0(\lambda) & \\ \hline & 0 & & I_p \end{array}\right]$$

such that $\Delta(\mathrm{GL}_p(A) \times \mathrm{GL}_p(A))$ commutes with $\rho(D^*)$. We are therefore reduced to the following situation: Let $G \subset \mathrm{GL}_{p,p}(A')$ be the subgroup of those

$$g = \begin{pmatrix} \alpha & x \\ 0 & \beta \end{pmatrix}$$

such that $\alpha \cdot \rho(\lambda) = \rho(\lambda) \cdot \alpha$ for all $\lambda \in D^*$ and let $G' \subset G$ be the subgroup where $x = 0$. We must show that $i_*: H_n(G'; k) \to H_n(G; k)$ is onto. G is the semi-direct product of G' and the abelian subgroup N of those g with $\alpha = \beta = 1$. If we let $\lambda \in D^*$ act on G by conjugation by $\rho(\lambda)$, then the action is trivial on G' and on N it is just the restriction of the D-module structure of N given by left multiplication on the rows.

Consider the action of D^* on the spectral sequence of the extension G of G' by N. By the lemma, $H_*(N; k)$ is a direct sum of eigenspaces associated to characters $D^* \to k^*$. Using the fact that D^* acts trivially on G' it follows that $E^2_{st} = H_s(G'; H_t(N; k))$, hence also E^r_{st} for $2 < r \leq \infty$, breaks up into eigenspaces preserved by the differentials. However, D^* acts trivially on $H_{s+t}(G; k)$ so the eigenspaces belonging to the trivial character form a spectral sequence

$$E^2_{s,t} = H_s(G'; H_t(N;k)^{D^*}) \Longrightarrow H_{s+t}(G;k).$$

By the lemma $H_t(N;k)$ has no non-trivial invariants for $0<t<d$ so G and G' have the same homology in degrees less than d.

It remains to prove the Lemma and we will need a well-known formula for the homology $H_*(N;k)$ of an abelian group N with coefficients in a field k. Let $p = \mathrm{char}(k)$, and let ${}_pN$ be the subgroup of elements of N killed by p if $p>0$, and ${}_pN = 0$ if $p=0$. Let $\Lambda(V)$ and $\Gamma(V)$ be the exterior and divided-power algebras respectively of a k-vector space V.

Proposition. *There exists an isomorphism of graded k-algebras*

$$(*) \qquad \Lambda(N \otimes_Z k) \otimes_k \Gamma({}_pN \otimes_Z k) \xrightarrow{\sim} H_*(N;k)$$

with $N \otimes_Z k$ of degree 1 and ${}_pN \otimes_Z k$ of degree 2. There is a canonical isomorphism, functorial in N, if either $p \neq 2$, or if $p=2$ and N is restricted to the full subcategory of abelian groups such that ${}_2N \subset 2N$. When $p=2$ and N is arbitrary, it is always possible to choose the isomorphism to be compatible with a given action of a finite group of odd order on N.

This result is contained for the most part in the Cartan seminar [4]. For the reader's convenience we shall now construct a homomorphism (*) with the required functorial properties. The fact that it is an isomorphism is proved by reducing to the case of cyclic groups and computing, and we refer to *loc. cit.* for the details. Recall that a "canonical" map or structure is always compatible with morphisms.

First of all, $H_*(N;k)$ has a canonical strictly anticommutative algebra structure with divided powers for elements of degree ≥ 2. The canonical isomorphism $N \otimes K = H_1(N,k)$, ($\otimes$ over $\mathbf{Z}$), extends to a canonical algebra homomorphism

$$\Lambda(N \otimes k) \to H_*(N;k).$$

When $\mathrm{char}(k)=0$, this is the map (*), so from now on we suppose $p>0$.

In a moment, we shall describe a canonical exact sequence

$$(**) \qquad 0 \to \Lambda^2(N \otimes k) \xrightarrow{i} H_2(N;k) \xrightarrow{j} {}_pN \otimes k \to 0$$

and show that it splits canonically if either p is odd or if $p=2$ and ${}_2N \subset 2N$, so that j has a canonical section s in these cases. On the other hand, if $p=2$ and N is endowed with an action of a group of odd order, the theorem of Maschke implies there exists a section s of j compatible with the given action. The section s extends uniquely to a homomorphism

$$\Gamma({}_pN \otimes k) \to H_*(N;k)$$

compatible with divided powers. Combining this with the above homomorphism from $\Lambda(N \otimes k)$, we obtain a homomorphism (*) with the functorial properties described in the proposition.

It remains to describe (**) and show its splits canonically under the indicated conditions. Since for any k-module V, we have

$$\mathrm{Ext}^1_Z(N, V) \cong \mathrm{Ext}^1_Z({}_pN, V) \cong \mathrm{Hom}_k({}_pN \otimes k, V)$$

$$H^2(N; V) \cong \mathrm{Hom}_k(H_2(N; k), V)$$

it suffices to describe a canonical exact sequence

$$0 \to \mathrm{Ext}^1_Z(N, V) \xrightarrow{j^*} H^2(N; V) \xrightarrow{i^*} \mathrm{Hom}_k(\Lambda^2 \bar{N}, V) \to 0,$$

($\bar{N} = N \otimes k$) splitting canonically under the indicated conditions. Elements of the Ext group (resp. $H^2(N; V)$) classify abelian group extensions (resp. central extensions) of N by V. Let j^* be the obvious inclusion and let i^* associate to a central extension its commutator pairing, i.e., the pairing obtained by lifting two elements of N into the extension and taking their commutator. As a central extension is abelian if and only if its commutator pairing is trivial, the sequence is exact except for the surjectivity of i^*.

To establish this point, suppose given a map $\Lambda^2\bar{N} \to V$, that is, an alternating Z-bilinear map $h : N \times N \to V$. Since k is a field, the map $\Lambda^2\bar{N} \to \bar{N} \otimes_k \bar{N}$ sending $x \wedge y$ to $x \otimes y - y \otimes x$ is injective, hence there is a Z-bilinear map $f : N \times N \to V$ such that

$$h(n, n') = f(n, n') - f(n', n).$$

Then f is a 2-cocycle, so the set $N \times V$ with the operation

$$(n, v)(n', v') = (nn', f(v, v') + v + v')$$

is a central extension of N by V. It is clear that the commutator pairing of this extension is h, so i^* is surjective.

When p is odd, there is a canonical choice for f:

$$f(n, n') = \tfrac{1}{2} h(n, n')$$

so i^* has a canonical section as claimed. (Note that i^* and this section are necessarily k-module homomorphisms, as they are morphisms of representable functors.) Suppose finally that $p = 2$ and ${}_2N \subset 2N$. The commutator pairing of a central extension of N by V vanishes on $2N$, hence on ${}_2N$, so the restriction of the extension to ${}_2N$ is abelian. This gives us a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_Z(N, V) \arrow[r] \arrow[d, "\cong"'] & H^2(N; V) \arrow[d] \arrow[dl, dashed] \\
\mathrm{Ext}^1_Z({}_2N, V) \arrow[r] & H^2({}_2N; V)
\end{tikzcd}$$

where the vertical arrows are restriction from N to ${}_2N$, showing the exact sequence splits canonically in this case as claimed.

Proof of the Lemma. First suppose $\mathrm{char}(k)=0$. We may assume that k is the algebraic closure of Q in C. Let F be a totally real number field of degree d; it exists, for by Dirichlet's theorem there is an odd prime number l such that d divides $\frac{1}{2}(l-1)$, so one can take F to be a subfield of $Q(\exp(2\pi i/l))$. We take D to be the ring of integers in F.

If N is a D-module, then $N\otimes_Z k$, as a representation of D^*, is a direct sum of copies of $F\otimes_Q k$ with D^* acting by multiplication on F. By Galois theory, there is a ring isomorphism: $F\otimes_Q k\cong k^d$ having for its components the homomorphisms $x\otimes y\to\sigma(x)y$, where σ runs over the d distinct embeddings of F in k. Thus as a representation of D^*, $N\otimes_Z k$ is a direct sum of one-dimensional representations with characters $\sigma: D^*\to k^*$, hence

$$H_t(N;k)=\Lambda^t(N\otimes_Z k)$$

is a direct sum of one-dimensional representations with characters $\prod\sigma^{n_\sigma}$, where the n_σ are integers $\geqslant 0$ such that $\sum n_\sigma=t$. Assume the family $\{n_\sigma\}$ is such that this character is trivial. Then

$$\sum n_\sigma\log|\sigma(u)|=0$$

for all u in D^*, where $|\quad|$ is the absolute value in C. By the Dirichlet unit theorem, this happens if and only if all the n are equal. Thus if $t>0$, all the n_σ are $\geqslant 1$ and $t\geqslant d$, showing that $H_t(N;k)$ does not contain the trivial representation for $0<t<d$. This proves the lemma when $\mathrm{char}(k)=0$.

Suppose now that $p=\mathrm{char}(k)>0$, and let k_d be the subfield of k with p^d elements. Since the norm $N: k_d^*\to F_p^*$ is surjective, its kernel is cyclic of order $(p^d-1)/(p-1)$; let x generate the kernel, and let $g(X)$ be the minimal polynomial of x over F_p. Note that $k_d=F_p(x)$, for if the latter field had degree $j<d$, then $(p^d-1)/(p-1)$ would divide p^j-1, which is impossible as the former number is $>p^j$. Hence $g(X)=X^d+b_1X^{d-1}+\cdots+b_d$ had degree d, and $b_d=(-1)^dNx=\pm 1$.

Let $f(X)=X^d+\cdots+a_d$, $a_i\in\mathbf{Z}$, $a_d=\pm 1$, reduce mod p to $g(X)$, and let $D=\mathbf{Z}[X]/(f(X))$. Note that f is irreducible as g is, hence D is an order in a number field of degree d. The image of X in D is invertible as $a_d=\pm 1$. Therefore we have an isomorphism $D/pD\cong k_d$, and the image of the character $\phi: D^*\to k_d^*$ induced by this isomorphism contains the cyclic subgroup of order $(p^d-1)/(p-1)$.

Let N be a D-module. We claim there is an isomorphism

$$\Lambda(N\otimes_Z k)\otimes_k\Gamma({}_pN\otimes_Z k)\xrightarrow{\sim}H_*(N,k)$$

commuting with the action of D^*. This is clear from the Proposition if p is odd, or if $p=2$ and ${}_2N\subset 2N$. On the other hand, if $p=2$ and $2N=0$, then N is a module over $D/2D = k_d$, hence D^* acts on N through the group k_d^* of odd order, and the proposition furnishes the required isomorphism in this case. In general when $p = 2$, we choose a complement N'' for the kernel (resp. Q for the image) of the homomorphism ${}_2N \to N/2N$ of k_d-vector spaces, and let N' be the inverse image of Q in N. Then $N \cong N' \oplus N''$ as D-modules, and ${}_2N' \subset 2N'$, $2N'' = 0$, so upon tensoring the isomorphisms obtained already for N' and N'', we obtain the required isomorphism for N.

As D^*-modules, N/pN and ${}_pN$ are direct sums of copies of the one-dimensional representation over k_d with character ϕ. By Galois theory, there is a ring isomorphism: $k_d \otimes_{\mathbf{Z}} k \cong k^d$ whose components are the homomorphisms $x \otimes y \to xP^a y$ for $0 \leqslant a < d$, hence as a representation of D^*, $k_d \otimes_{\mathbf{Z}} k$ is the direct sum of the one-dimensional representations with character ϕ^{p^a} for $0 \leqslant a < d$. It follows that we have D^*-isomorphisms

$$N \otimes_{\mathbf{Z}} k \cong \oplus V_a \otimes L^{\otimes p^a}, \qquad {}_pN \otimes_{\mathbf{Z}} k \cong \oplus W_a \otimes L^{\otimes p^a}$$

the sum being over $0 \leqslant a < d$, where L is the one-dimensional representation with character ϕ, and where D^* acts trivially on V_a and W_a. Thus

$$\begin{aligned} \Lambda(N \otimes_{\mathbf{Z}} k) \otimes \Gamma({}_pN \otimes_{\mathbf{Z}} k) &\cong \bigotimes_a [\Lambda(V_a \otimes L^{\otimes p^a}) \otimes \Gamma(W_a \otimes L^{\otimes p^a})] \\ &\cong \bigoplus_{\{m_a, n_a\}} \left[\bigotimes_a (\Lambda^{m_a}(V_a) \otimes \Gamma^{n_a}(W_a)) \right] \otimes L^{\otimes \Sigma (m_a + n_a) p^a} \end{aligned}$$

where the sum is taken over the families of non-negative integers m_a, n_a for $0 \leqslant a < d$. This shows that $H_*(N, k)$ is a direct sum of one-dimensional representations of D^* as claimed.

It remains to show that the trivial representation does not occur in degrees t with $0 < t < d$. The direct summand corresponding to $\{m_a, n_a\}$ is homogeneous of degree $\Sigma(m_a + 2n_a)$, and if this summand contains the trivial representation, then

$$\sum_{0 \leqslant a < d} (m_a + m_a)p^a \equiv 0 \quad \mathrm{mod}(p^d - 1)/(p - 1)$$

because by construction $\phi(D^*)$ contains the cyclic subgroup of k^* of this order. Let $\{m_a, n_a\}$ be a non-zero solution of this congruence such that $t = \Sigma(n_a + 2n_a)$ is minimal. Clearly $n_a = 0$ for all a. If $m_b \geqslant p$ for some b, then by replacing m_b by $m_b - p$ and m_{b+1} by $m_{b+1} + 1$, (or m_0 by $m_0 + 1$ if $b = d - 1$), and keeping the others the same, we would get a new solution contradicting minimality. Thus $m_a < p$ for $0 \leqslant a < d$, so by the uniqueness of the p-adic expansion, we see that the minimal solution is $m_a = 1$, $n_a = 0$ for all a. The minimal degree t is d, so the proof of the lemma is complete.

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